Linear Algebra II

04/03/2019, Monday, 19:00 – 21:00

1 $(6+7+7=20$ pts) **Inner product spaces**

Let V be a real inner product space with inner product denoted by $\langle v, w\rangle$. Let $\|v\|$ denote the norm associated with this inner product.

(a) Show that Pythagoras' theorem holds, i.e. if $v, w \in V$ are orthogonal then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.

(b) Let $S \subset V$ be an n-dimensional subspace of V and let $v \in V$. Let p be the orthogonal projection of v onto S. Show that $\|p\| \leqslant \|v\|$. Under what condition do we have equality here?

(c) Also show that $\|v-p\| \leqslant \|v\|$. Show that equality holds if and only if v is orthogonal to S.

2 $(4+4+4+8=20$ pts) **Least squares approximation**

Consider the inner product space $C[-1,1]$ with inner product

$$(f,g) := \int_{-1}^{1} f(x)g(x)dx.$$

Let $\mathcal{S}$ be the subspace of all functions of the form $g(x) = a + bx$ with $a, b \in \mathbb{R}$.

(a) Show that the functions 1 and x are orthogonal.

(b) Compute $\|1\|$ and $\|x\|$.

(c) Determine an orthonormal basis of $\mathcal{S}$.

(d) Compute the best least squares approximation of the function $f(x) = x^{\frac{1}{3}}$ by a function from the subspace $\mathcal{S}$.

3 (6 + 6 + 6 + 7 = 25 pts) **Diagonalization**

Let A be a real $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_n$.

(a) Prove that $\det(A)$ is equal to the product $\lambda_1 \lambda_2 \ldots \lambda_n$ of the eigenvalues.

(b) Prove that A^{-1} exists if and only if all eigenvalues of A are nonzero.

(c) Assume that A^{-1} exists. Prove that A^{-1} is diagonalizable if and only if A is diagonalizable.

(d) Assume that A is symmetric and that A^{-1} exists. Prove that A is unitarily diagonalizable if and only if A^{-1} is unitarily diagonalizable.

4 (6 + 6 + 7 + 6 = 25 pts) **Hermitian matrices**

A matrix $A \in \mathbb{C}^{n \times n}$ is called normal if $A^H A = AA^H$. A complex matrix is called skew-Hermitian if $A^H = -A$. In this problem, you may use the following result from the book: a complex matrix is normal if and only if it is unitarily diagonalizable.

(a) Show that if $A \in \mathbb{C}^{n \times n}$ is Hermitian, then it is normal and all its eigenvalues are real.

(b) Show that if $A \in \mathbb{C}^{n \times n}$ is normal and all its eigenvalues are real, then A is Hermitian.

(c) Show that if $A \in \mathbb{C}^{n \times n}$ is skew-Hermitian, then it is normal and all its eigenvalues lie on the imaginary axis, i.e. for every eigenvalue λ of A we have $\mathrm{Re}(\lambda) = 0$.

(d) Show that if $A \in \mathbb{C}^{n \times n}$ is normal and all its eigenvalues lie on the imaginary axis, then it is skew-Hermitian.

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10 pts free